Particular Solutions to non-homogeneous equations

$$y = c_1y_1(t) + c_2y_2(t) + Y(t)$$

TABLE 3.6.1 The Particular Solution of $ay'' + by' + cy = g_i(t)$

$g_i(t)$	$Y_i(t)$
$P_n(t) = a_0t^n + a_1t^{n-1} + \cdots + a_n$	$t^s(A_0t^n + A_1t^{n-1} + \cdots + A_n)$
$P_n(t)e^{\alpha t}$	$t^s(A_0t^n + A_1t^{n-1} + \cdots + A_n)e^{\alpha t}$
$P_n(t)e^{\alpha t}\begin{cases}\sin \beta t\\ \cos \beta t\end{cases}$	$t^s[(A_0t^n + A_1t^{n-1} + \cdots + A_n)e^{\alpha t}\cos \beta t + (B_0t^n + B_1t^{n-1} + \cdots + B_n)e^{\alpha t}\sin \beta t]$

Notes. Here s is the smallest nonnegative integer ($s = 0$, 1, or 2) that will ensure that no term in $Y_i(t)$ is a solution of the corresponding homogeneous equation. Equivalently, for the three cases, s is the number of times 0 is a root of the characteristic equation, α is a root of the characteristic equation, and $\alpha + i\beta$ is a root of the characteristic equation, respectively.

Variation of Parameters

If the functions p, q, and g are continuous on an open interval I, and if the functions y_1 and y_2 are linearly independent solutions of the homogeneous equation (18) corresponding to the nonhomogeneous equation (16),

$$y'' + p(t)y' + q(t)y = g(t),$$

then a particular solution of Eq. (16) is

$$Y(t) = -y_1(t)\int \frac{y_2(t)g(t)}{W(y_1,y_2)(t)}\,dt + y_2(t)\int \frac{y_1(t)g(t)}{W(y_1,y_2)(t)}\,dt, \qquad (28)$$

and the general solution is

$$y = c_1y_1(t) + c_2y_2(t) + Y(t), \qquad (29)$$

as prescribed by Theorem 3.6.2.

Derivatives

Sin u = Cos u * u'

Cos u = –Sin u * u'

Tan u = $\sec^2$u * u'

Sec u = (Sec u * Tan u)u'

Csc u = – (csc u * cot u)u'

Cot u = – ($\csc^2$u)u'

Arcsin u = u' / $(1 – u^2)^{1/2}$

Arccos u = – u' / $(1 – u^2)^{1/2}$

Arctan u = u' / 1+ u^2

Arcsec u = u' / |u| * $(u^2 – 1)^{1/2}$

Arccsc u = – u' / |u| * $(u^2 – 1)^{1/2}$

Arccot u = – u' / 1 + u^2

Integrals

Sin u = – cos u + C

Cos u = sin u + C

Tan u = – ln |cos u| + C

Sec u = ln |sec u + tan u| + C

Csc u = – ln |csc u + cot u| + C

Cot u = ln |sin u| + C

Sec^2u = tan u + C

Csc^2u = – cot u + C

Sec u * Tan u = sec u + C

Csc u * Cot u = – csc u + C

du / $(a^2 – u^2)^{1/2}$ = arcsin (u/a) + C

du / $a^2 + u^2$ = (1/a) arctan (u/a) + C

du / $u(u^2 – a^2)^{1/2}$ = (1/a) arcsec (|u|/a) + C

Power Reducing Formulas

Sin^2u = (1 – cos2u)/2

Cos^2u = (1 + cos2u)/2

Tan^2u = (1 – cos2u)/(1 + cos2u)

Double Angle Formulas

Sin2u = 2sinucosu

Cos2u = $\cos^2$u – $\sin^2$u

= $2\cos^2$u – 1 = 1 – $2\sin^2$u

Tan2u = (2tan u) / (1 – $\tan^2$u)

Strange Integrals

Fnint(tan x) = ln |sec x|

Fnint(sec x) = ln |sec x + tan x|

Strange Identities

Sin A Cos B = (sin(A - B) + sin(A + B))/2

Sin A Sin B = (cos(A - B) - cos(A + B))/2

Cos A Cos B = (cos(A - B) + cos(A + B))/2

Sin^2x + $\cos^2$x = 1

1 + $\tan^2$x = $\sec^2$x

Sec^2x – 1 = tan x

Wronskian Stuff

$W = \begin{vmatrix} e^{r_1t} & e^{r_2t} \\ r_1e^{r_1t} & r_2e^{r2t} \end{vmatrix} \neq 0$

if W = 0 on interval I, then f is linearly dependent for all numbers on I

if W != 0 on interval I, then f is linearly independent for all numbers on I

Given y'' + p(t) y' + q(t) y = 0, $\quad W(y_1, y_2)(t) = c \int –\exp[- p(t)dt]$

Different Types of Differential Equations

1. First Order Equations
   - $y' + p(t)\,y = g(t)$
   - $\mu = e^{\int p(t)dt}$
   - $y = \int \mu g/\mu$
2. Separable Equations
   - $M(x) + N(y)y' = 0$
   - $\int Mx)\,dx = –\int N(y)dy$
3. Exact Equations
   - $M(x, y) + N(x, y)y' = 0$
   - Find ψ such that $\psi_x = M$, $\psi_y = N$
   - ψ exists *iff* $M_y(x, y) = N_x(x, y)$
4. Integrating factors
   - Depending on both x and y
     - $d\mu/dx = (M_y – N_x) / N * \mu$
   - Depending just on y
     - $\mu = \int \exp Q(y)\,dy$
5. $b^2 – 4ac > 0$ $\quad r_1 != r_2$
   - Use quadratic equation to find roots
   - $y_1 = c_1e^{r1t}$
   - $y2 = c_2e^{r2t}$
   - $y = c_1e^{r1t} + c_2e^{r2t}$
6. $b_2 – 4ac < 0$ $\quad$ r1 != r2
   - μ = Rad (4ac – b2) / 2a
   - $\lambda = –b/2a$
   - $r_1 = \lambda + i\mu$
   - $r_2 = \lambda – i\mu$
   - $y_1 = e^{\lambda t}\cos(\mu t)$
   - $y_2 = e^{\lambda t}\sin(\mu t)$
   - $y = e^{\lambda t}\cos(\mu t) + e^{\lambda t}\sin(\mu t)$
7. b2 – 4ac = 0 $\quad$ r1 = r2 = –b/2a
   - $y_1 = e^{(–b/2a)\,t}$
   - $y_2 = t\,e^{(–b/2a)\,t}$
   - $y = e^{(–b/2a)\,t} + t\,e^{(–b/2a)\,t}$

Reduction of Order

$y'' + p(t)\,y' + q(t)\,y = 0$

$y = v(t) * y_1(t)$

$y' = v'(t) * y_1(t) + v(t) * y'_1(t)$

$y'' = v''(t) * y_1(t) + 2v'(t) * y_1'(t) + v(t) * y_1''(t)$

$y_1 * v'' + (2y_1' + py_1)v' + (y_1'' + py_1' + qy_1)v = 0$

$y_1 * v'' + (2y_1' + py_1)v' = 0$

Solve this 1st order equation and integrate to find the 2nd solution