Math Notes – Multivariable Calculus

These notes covers parametric equations, polar coordinates, vectors, planes and lines in 3D, spherical and cylindrical coordinates, vector functions and space curves, and multivariable differentiation and integration. They're based on James Stewart's Calculus Early Transcendentals Fourth Edition, UCSD Math 21C

Stop taking shortcuts

I. Parametric Equations

- a. Eliminate parameter
- b. Graph by plotting points
- **c. Tangents and derivatives of Parametric Equations**
  - i. Take dy/dx to get the slope, use point slope form w/ a point
  - ii. If dy = 0, the tangent is horizontal
  - iii. If dx = 0, the tangent is vertical
  - iv. 2nd derivative = (d (dy/dx)) / dx
- **d. Areas of Parametric Equations**
  - i. Area = FnInt (y dx, t, a, b)
  - ii. The y value is from the x axis, so adjust if necessary
  - iii. Make sure to use dx
- **e. Arc Length of Parametric Equations**
  - **i. L = FnInt (Rad ($(dx / dt)^2 + (dy / dt)^2$), t, a, b)**
  - ii. 1 comes from FnInt (Rad ($1 + (dy / dx)^2$), x, a, b)
  - iii. Arc length does not increase for multiple loops (make sure the interval doesn't repeat)
- **f. Surface Area of Parametric Equations**
  - **i. S = FnInt (2 * π * y * Rad ($1 + (dy / dx)^2$), x, a, b)**
  - ii. If integrating w/ respect to x (rotating about y axis), use x instead of y

II. Polar Coordinates

- **a. Equations to convert from polar to rectangular and vice versa**
  - i. x = r cosθ
  - ii. y = r sinθ
  - iii. $r^2 = x^2 + y^2$
  - iv. tanθ = y/x
- **b. Common shapes**
  - i. θ = x – line
  - ii. r = x – circle of radius x, centered at origin
  - iii. r = x cosθ – circle, left side touching origin, diameter of x, radius of x/2
  - iv. r = x sinθ – circle, bottom touching origin, diameter of x, radius of x/2
- c. It is often useful to complete the square when converting from polar – rectangular
- **d. Tangents to polar curves**
  - i. Write the curve as a parametric equation
    1. x = r cosθ = f(θ) cosθ (the polar equation)
    2. y = r sinθ = f(θ) sinθ
  - ii. Take dy/dx, which equals the slope of the tangent line
    1. Officially, dy/dx = (dr/dθ * sinθ + r cosθ) / (dr/d θ * cosθ – r sinθ)
  - iii. In order to find horizontal and vertical tangents, set dy/dθ or dx/dθ to 0, respectively
  - iv. Tangents at the pole - trick
    1. dy/dx = tanθ if dr/dθ != 0
- **e. Area in Polar Coordinates**
  - i. A = FnInt (1/2 * $[f(\theta)]^2$, θ, a, b)
  - ii. A = FnInt(1/2 * r^2, θ, a, b)
  - **iii. Don't forget the ½**
  - iv. When integrating, the area sweeps from the angle line to the poles
  - v. When finding area inside a curve and outside another, subtract
  - vi. When finding common area, add
- **f. Arc Length in Polar Coordinates**
  - i. L = FnInt(Rad ($r^2 + (dr/d\theta)^2$), x, a, b)
  - ii. Do NOT complete the square
  - iii. Rad 4 + Rad 6 != Rad 10

III. Vectors

- a. Given two points, $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, Vector **a** = $<x_2 – x_1, y_2 – y_1, z_2 – z_1>$
- b. |a| - length of vector A = Rad $(a_1^2 + a_2^2)$
- c. Adding vectors – add the components
- d. Subtracting vectors – add the negative vector
- e. Multiplying by a scalar – multiply all the components by the scalar
- f. Know triangle and parallelogram laws for adding vectors
- g. All properties of numbers (cumulative, associative, distributive) apply to vectors also
- h. Standard Basis Vectors – i <1, 0, 0>, j <0, 1, 0>, k <0, 0, 1>
- i. Unit Vector = **a** / |a|
- j. Two vectors are parallel if they’re scalar multiples of each other
- **k. Dot Product**
  - **i. The dot product of two vectors returns a scalar**
  - ii. <a1, b1, c1> * <a2, b2, c2> = $a_1a_2 + b_1b_2 + c_1c_2$
  - iii. **a** * **b** = |**a**| |**b**| cosθ
  - iv. cosθ= (**a** * **b**) / (|**a**| |**b**|)
  - v. If **a** * **b** = 0, the vectors are orthogonal (perpendicular)
- **l. Direction Angles**
  - i. The direction angles of **a** are the angles α, β, and γ on $[0, \pi]$ that a makes with the positive x, y, and z axes
- **m. Direction Cosines**
  - i. The cosine values of the direction angles
  - ii. The direction cosines of **a** are the components of the unit vector in the direction of **a**
  - iii. $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$
  - iv. Derivation
    - 1. Cos α = **a** * **i** / |**a**| |**i**| = a_1 / |**a**|
    - 2. Cos β = a_2 / |**a**|
    - 3. Cos γ = a_3 / |**a**|
    - 4. **a** = <a1, a1, a3> = < |**a**| cos α, |**a**| cos β, |**a**| cos γ>
    - 5. **a = |a|** <cos α, cos β, cos γ>
    - 6. **a / |a| =** <cos α, cos β, cos γ>
- n. **Scalar Projections**
  - i. Projection of **b** onto **a** – **a** * **b** / |**a**|
- **o. Vector Projections**
  - i. The scalar projection times the unit vector in the direction of **a**
  - ii. (**a** * **b**) * **a**/|**a**|^2
  - iii. **a** * **b** returns a scaler, which is then multiplied by **a**
- **p. Cross Product**
  - **i. The cross product of two vectors returns another vector**
    - **1. It is illegal to cross a vector w/ a scaler**
  - ii. Use a determinant to determine cross products
    - **1. Remember to use opposite signs when evaluating determinants**
    - **2. Don’t’ take shortcuts, write out values even if they’re 0**
  - iii. **a** x **b** = $\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$
  - iv. | **a** x **b** | = |**a**| |**b**| sinθ
  - v. The vector **a** x **b** is orthogonal to both **a** and **b**
  - vi. If the cross product = 0, then the vectors are parallel (sin 0 = 0)
  - vii. | **a** x **b** | = the area of the parallelogram determined by **a** and **b**
  - viii. Right Hand Rule – thumb points in the direction of the resulting vector
  - ix. The cumulative and associative properties do NOT work w/ cross products
    - 1. **a** x **b** = - **b** x **a**
    - 2. (*c***a**) * **b** = *c*(**a** x **b**) = **a** x (*c***b**)
    - 3. **a** x (**b** + **c**) = **a** x **b** + **a** x **c**
    - 4. **a** * (**b** x **c**) = (**a** x **b**) * **c**
    - 5. **a** x (**b** x **c**) = (**a** * **c**)**b** – (**a** * **b**)**c**
- **q. Scalar Triple Product**
  - i. **a** * (**b** x **c**) = $\begin{vmatrix} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \end{vmatrix}$

- ii. The scalier triple product is the volume of the parallelepiped determined by **a**, **b**, **c**
- iii. Take the absolute value if negative
- iv. If the scalar triple product is 0, then the vectors are coplaner

IV. Planes, Lines, and Surfaces in Three Dimensional Space

- a. Distance Formula – Rad $((x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2))$
- b. Equation of a sphere – $(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$
- **c. Lines**
  - i. A line is defined by a point and a direction\
  - ii. **Parallel Lines** – lines in the same plane that never intersect
  - iii. **Intersecting Lines** – lines that intersect (duh)
  - iv. **Skew Lines** – lines that are not parallel but don't intersect, meaning they're not in the same plane
    - 1. Check for parallel lines first
    - 2. If two lines are not parallel (different direction vectors), check if they intersect
    - 3. If they're neither parallel nor intersect, they must be skew
  - **v. Vector equation of a line in 3D space**
    - 1. $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, *t* is any scalar multiple
    - 2. **r** is the distance from the origin to a point (x, y, z) on the line
    - 3. $\mathbf{r}_0$ is the vector from the origin to an initial point (x_0, y_0, z_0) on the line
    - 4. **v** is a vector parallel to the line
  - **vi. Parametric Equations of the line**
    - 1. $x = x_0 + at$
    - 2. $y = y_0 + bt$
    - 3. $z = z_0 + ct$
    - 4. Derivation
      - a. **v** can also be written as <a, b, c>, so $t\mathbf{v} = \langle ta, tb, tc\rangle$
      - b. $\mathbf{r} = \langle x, y, z\rangle$, $r_0 = \langle x_0, y_0, z_0\rangle$
      - c. $\langle x, y, z\rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc\rangle$
  - **vii. Symmetric Equations of the line**
      - a. $(x - x_0) / a = (y - y_0) / b = (z - z_0) / c$
- **d. Planes**
  - i. A plane is defined as a point in the plane and a normal vector to the plane
  - ii. Try to find two vectors in a plane, which you can cross to get a normal vector
  - iii. Two planes are perpendicular if their normal vector cross product is 0, and parallel if their normal vectors are scalar multiples of each other
  - **iv. Vector Equation of a plane in 3D space**
    - 1. A plane is defined as a vector in the plane and a normal vector to the plane
    - 2. A normal vector **n** is a vector that is perpendicular to the plane at all points
    - 3. Given 2 points, P (x, y, z) and P_0 (x_0, y_0, z_0), $n * (r - r_0) = 0$
    - 4. $n * r = n * r_0$
  - **v. Scalar Equations for the Plane**
    - 1. $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$
    - 2. Derivation
      - a. $n = \langle a, b, c\rangle$
      - b. $r = \langle x, y, z\rangle$
      - c. $r_0 = \langle x_0, y_0, z_0\rangle$
    - 3. $\langle a, b, c\rangle * \langle x - x_0, y - y_0, z - z_0\rangle = 0 \Rightarrow a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$
  - **vi. Linear Equation of a Plane**
    - 1. $ax + by + cz + d = 0$
    - 2. This plane has a normal vector **n** <a, b, c>
  - **vii. Distance from a point to a plane**
    - 1. P_1 (x_1, y_1, z_1) to the plane $ax + by + cz + d = 0$
    - 2. $|ax_1 + by_1 + cz_1 + d|$ / Rad $(a^2 + b^2 + c^2)$
    - 3. Derivation
      - a. Let $P_0 = (x_0 + y0 + z_0)$ – any point in the given plane
      - b. **b** = P_0 to $P_1 = \langle x - x_0, y - y_0, z - z_0\rangle$
      - c. The distance is the scalar projection of **b** onto the normal vector **n**
- **e. Cylinders**
  - i. **Traces** – the curves of intersection of the surface w/ planes parallel to the coordinate planes
  - ii. **Cylinder** – a surface that consists of all lines that are parallel to a given line and pass through a given plane curve
  - iii. $z = x^2$ is a parabolic cylinder

- iv. $x^2 + y^2 = 1$ is a cylinder parallel to the z axis

- **f. Quadric Surfaces**
  - **i. Standard forms**
    - 1. $Ax^2 + By^2 + Cz^2 + J = 0$
    - 2. $Ax^2 + By^2 + Iz = 0$
  - ii. To sketch, set each variable equal to k and 0 to determine how the shape looks in each direction
  - iii. Make sure to set limits for k, so that the shape still makes sense (ie. No negative radius spheres)
  - **iv. All shifting rules apply**
  - **v. Forms**
    - 1. Ellipsoid
      - a. $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$
      - b. All traces are ellipse
    - 2. Elliptic Paraboloid
      - a. $z/c = x^2/a^2 + y^2/b^2$
      - b. Horizontal traces are ellipses, Vertical traces are parabolas
      - c. The variable to the 1st power indicates the axis of the paraboloid
      - d. If a and b are equal, then it's a circular parabolid
    - 3. Hyperbolic Paraboloid
      - a. $z/c = x^2/a^2 - y^2/b^2$
      - b. Horizontal traces are hyperbolas, vertical traces are parabolas
      - c. If $c < 0$, it has the shape of an upward parabola. If $c > 0$, downward parabola
    - 4. Cone
      - a. $z^2/c^2 = x^2/a^2 + y^2/b^2$
      - b. Horizontal traces are ellipses
      - c. Vertical traces are hyperbolas if $x = y = k$, lines are $x = y = 0$
    - 5. Hyperboloid of One Sheet
      - a. $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$
      - b. Horizontal traces are ellipses, Vertical traces are hyperbolas
      - c. Axis is symmetry is the variable that's negative
    - 6. Hyperboloid of two sheets
      - a. $-x2/a2 - y2/b2 + z2/c2 = 1$
      - b. Horizontal traces are ellipses, Vertical traces are hyperbolas
      - c. The two minus signs indicate two sheets
    - 7. Parabolic cylinder
      - a. $z = x^2$
      - b. Horizontal Traces are pairs of lines
      - c. Vertical traces are parabolas
      - d. Missing variable is axis

V. Cylindrical Coordinates

- a. Polar coordinates in 3D. Use polar for the x/y plane, and move it up z units into 3D space
- b. Use for describing spheres and other easily polar defined shapes
- c. Takes the format (r, θ, z)
- **d. Converting from rectangular to cylindrical and vice versa**
  - i. $x = r \cos\theta$
  - ii. $y = r \sin\theta$
  - iii. $z = z$
  - iv. $r^2 = x^2 + y^2$
  - v. $\tan\theta = y./x$
  - vi. $z = z$
  - **vii. Make sure θ is in the right quadrant**

VI. Spherical Coordinates

- a. Takes the format (ρ, θ, ϕ)
- b. Very useful for describing shapes that are symmetric about a point, w/ is usually defined as the origin
- c. ρ – the distance from the origin to the point
- d. θ - angle from the x axis to the projection of ρ in the xy plane
  - i. It's also the same θ in cylindrical coordinates
- e. ϕ – the angle from the z axis to $\rho - 0 < \phi < \pi$
- f. ϕ swings in the way theta swings, but it has to be positive and between 0 and π
- g.

h. **Converting from Spherical to rectangular and vice versa**
  i. $x = \rho \sin\phi \cos\theta$
  ii. $y = \rho \sin\phi \sin\theta$
  iii. $z = \rho \cos\phi$
  iv. $\rho = x^2 + y^2 + z^2$
i. **Common Forms**
  i. $\rho = c$ – a sphere
  ii. $\theta = c$ – a half plane – half because ϕ can never be negative
  iii. $\phi = c$, $0 < c < \pi/2$ – a half cone
  iv. $\phi = c$, $\pi/2 < c < \pi$ – half cone, opening downwards

VII. Vector Functions and Curves

a. A vector function is a function that has real numbers as its domain and vectors as its range
b. For every real value t, there is a unique vector <f(t), g(t), h(t)>, or f(t)**i**, g(t)**j**, h(t)**k**
c. The limit of a vector function is defined by taking the limits of its component functions
  i. The limit of a vector function returns another vector
d. A vector is continuous at *a* if and only if all its component functions are continuous at *a*
e. Vector functions that are in terms of linear functions of t are lines, and can be written as $r = r_0 + tv$
f. **Space Curves**
  i. x = f(t)
  ii. y = g(t)
  iii. z = h(t)
  iv. A curve defined by the three parametric equations w/ the position vector **r***(t)* = <f(t), g(t), h(t*)*> going from the origin to any point on the curve
  v. In order to draw a space curve, reduce it to two dimensions, get an projection, then scale it to 3D
  vi. In order to write a vector function in parametric form, reduce it to two dimensions, then choose one variable to be the parameter, and write everything off of that.
  vii. A helix consists of a cylinder, cone, or similar shape w/ one variable being set to the parameter t
g. **Derivatives of Vector Functions**
  i. Definition
    1. dr/dt = **r**'(t) = lim ($h \to 0$) (r(t + h) – r(t))/h
  ii. If **r**(t) = <f(t), g(t), h(t)> = f(t)**i** + g(t)**j** + h(t)**k**, **r**'(t) = <f'(t), g'(t), h'(t)> = f'(t)**i** + g'(t)**j** + h'(t)**k**
  iii. The 2nd derivative is the derivative of the 1st derivative
  iv. **Smooth curves** – a curve is smooth if **r**' is continuous and |**r**'(t)| != 0 except at the endpoints
  v. **Cusp** – any sharp change in behavior. Curves that contain cusps are NOT smooth
  vi. **Piecewise Smooth** – a curve that is made up of a finite number of smooth pieces
h. **Tangents to Vector Functions**
  i. Definition – the tangent line to the curve at any point is defined to bhe the line through the point parallel to the tangent vector **r**'(t)
  ii. **Unit Tangent Vector** = **T**(t) = **r**'(t) / |**r**'(t)|
  iii. **Finding the tangent line to a curve**
    1. Find the value of t at the initial point
    2. Take the derivative of the curve
    3. Evaluate the derivate for t
    4. For parametric form, be sure to obtain the initial point and the derivative vector <a, b, c>
      a. $x = x_0 + at$
      b. $y = y_0 + yt$
      c. $z = z_0 + zt$
i. **Integrals of Vector Functions**
  i. The integral of **r**(t)dt is the integral of the invididual componets of **r**
j. **Arc Length of Vector Functions**
  i. L = FnInt (Rad $((dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2)$, t, a, b)
  ii. L = FnInt (|**r**'(t)dt|, t, a, b)
  iii. **Parameterize a Curve with Respect to Arc Length**
    1. s = s(t) = FnInt (| **r**'(u)du |, u, 0, t)
    2. | r'(u) du | = Rad $((dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2)$ = ds/dt
    3. Solve for t, substitute back into original for t
k. **Curvature**
  i. **k (t) = |r'(t) x r''(t)| / |r'(t)|^3**
  ii. Other General Forms
    1. k(t) = |dT/ds|
    2. k(t) = |T'(t)/r'(t)|

- iii. **Curvature of a Plane Curve $y = f(x)$**
  - 1. $k(x) = |f''(x)| / [1 + (f'(x))^2]^{3/2}$
- iv. **Principle Unit Normal Vector**
  - 1. $N(t) = T'(t) / |T'(t)|$
  - 2. The unit normal vector is orthogonal to the tangent to both Tangent and Binormal Vectors
- v. **Binormal Vector**
  - 1. $B(t) = T(t) \times N(t)$
  - 2. It is perpendicular to both T and N, and is also a unit vector

VIII. Functions of Several Variables

- a. For every pair of numbers (x, y), there is a corresponding real number
- b. **Graphing**
  - i. Plot points, use level curves, reduce to 2D, etc
    - 1. **Planes** – set one variable to 0 and plot each 2D plane separately
  - ii. Level curves/Contour maps are curves w/ the equations $f(x, y) = k$
  - iii. To draw contour maps set the function equal to k and plot
- c. **Limits**
  - i. Approach the point of the limit from multiple angles. If the limit exists, the limit should be the same from all angles. If the limit approaching (0,0,0) on the x axis is different from the limit approaching (0,0,0) on the y axis, then the limit does NOT exist
  - ii. Test the axis first – set x, y = to zero respectively
  - iii. If the axis all yield the same limits, test lines $y = x$, and if that works, try the general line $y = mx$
  - iv. If all tests pass, then the limit exists
  - v. Remember that you cannot easily prove a limit exists, only that a limit does NOT exist
  - vi. If you end up with an interdeterminant form, L'Hospital's rule still works
- d. **Continuity**
  - i. f is continuous on D if f is continuous at every point (a, b) in D
  - ii. When determining continuity at (x, y, z), take the limit at (x, y, z) and see if it equals $f(x, y, z)$

IX. Partial Derivatives

- a. Definition
  - i. $f_x (a, b) = \lim (h \to 0) (f(a + h, b) - f (a, b)) / h$
  - ii. $f_y (a, b) = \lim (h \to 0) (f(a, b + h) - f (a, b)) / h$
- b. Notation
  - i. $f_x(x, y) = f_x = df/dx = d/dx\, f(x, y) = dz/dx$
  - ii. $f_y(x, y) = f_y = df/dy = d/dy\, f(x,y) = dz/dy$
- c. **Finding partial derivatives**
  - i. Differentiate w/ respect to one variable, assuming the other is a constant
  - ii. to find 2nd order derivatives, take the derivative again
  - iii. f_x and $f_y \to f_{xx}, f_{xy}, f_{yx}, f_{yy}$. $f_{xy} = f_{yx}$, assuming both are continuous
- d. **Tangent Lines w/ Multivariable Functions**
  - i. **Equation of a tangent plane to surface z = f(x, y) at point P (x0, y0, z0)**
    - 1. $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$
- e. **Linear Approximations**
  - i. Find the tangent plane, plug in new values into tangent plane
  - ii. **Given a point (x_0, y_0), the linear approximation is:**
    - 1. $L(x, y) = f(x, y) + f_x(x, y)(x - x_0) + f_y(x, y)(y - y_0)$
    - 2. Substitute the new values into the answer to the above equation
  - iii. **Total Differential**
    - 1. $dz = f_x(x, y)dx + f_y(x, y)dy\backslash$
    - 2. $\Delta Z = f$ (final values) – f (initial values)
- f. **Chain Rule**
  - i. If x and y are functions of a single variable t
    - 1. $dz/dt = f_x(x, y)(dx/dt) + f_y(x, y)(dy/dt)$
    - 2. $dz/dt = (dz/dx)(dx/dt) + (dz/dy)(dy/dt)$
    - 3. Equation 1 is the same as equation 2
  - ii. If x and y are functions of two variables s, t
    - 1. $dz/ds = f_x(x, y)(dx/ds) + f_y(x, y)(dy/ds)$
    - 2. $dz/dt = f_x(x, y)(dx/dt) + f_y(x, y)(dy/dt)$

    - 3. $dz/ds = (dz/dx)(dx/ds) + (dz/dy)(dy/ds)$
    - 4. $dz/dt = (dz/dx)(dx/dt) + (dz/dy)(dy/dt)$

- 5. Equations 1, 2 and 3, 4 are the same

- g. **Implicit Differentiation**
  - i. $dy/dx = -(dF/dx) / (dF/dy)$
  - ii. $dy/dx = -F_x / F_y$
  - iii. Equations 1 and 2 are the same

X. Directional Derivatives, Gradient Vector, Max/Min values, LaRange Multipliers

- a. **Gradient Vector**
  - i. $f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle = f_x\mathbf{i} + f_y\mathbf{j}$
- b. **Directional Derivatives**
  - i. The directional derivative is the rate of change in a single direction of a function
  - ii. The directional derivative is the dot product of two vectors and returns a **scalar** value
  - **iii. Make sure the direction vector that you're crossing w/ the gradient vector is a UNIT vector**
  - iv. **Functions of two variables**
    - 1. $D_uf(x, y) = \nabla f(x, y) * \mathbf{u}$
    - 2. Definition
      - a. Given a point (x_0, y_0) w/ a directional unit vector $u = \langle a, b\rangle$:
        $D_uf(x_0, y_0) = \lim (h \rightarrow 0)\ (f(x_0 + ha, y_0 + hb) - f(x_0, y_0)) / h$
    - 3. Given any unit vector $\mathbf{u} = \langle a, b\rangle$, $D_uf(x, y) = f_x(x, y)a + f_y(x, y)b$
  - v. **Functions of three Variables**
    - 1. $D_uf(x, y, z) = \nabla f(x, y, z) * \mathbf{u}$
    - 2. Definition
      - a. Given a point (x_0, y_0, z_0) w/ a directional unit vector $u = \langle a, b, c\rangle$:
        $D_uf(x_0, y_0, z_0) = \lim (h \rightarrow 0)\ (f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)) / h$
    - 3. Given any unit vector $\mathbf{u} = \langle a, b, c\rangle$, $D_uf(x, y, z) = f_x(x, y, z)a + f_y(x, y, z)b + fz(x, y, z)c$
  - vi. **Maximizing the Directional Derivative**
    - 1. The maximum value of the directional derivative is $|\nabla f(x)|$ and it occurs when u has the same direction as the gradient vector $\nabla f(x)$
  - vii. **Tangent Planes to Level Surfaces**
    - 1. $F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0)$
    - 2. The gradient vector $\nabla F(x_0, y_0, z_0)$ is perpendicular to the tangent vector and the plane itself
    - 3. The normal line passes through (x0, y0, z0) and is in the same direction as the gradient vector, w/ equations $(x - x_0)/F_x(x_0, y_0, z_0) = (y - y_0)/F_x(x_0, y_0, z_0) = (z - z_0)/F_x(x_0, y_0, z_0)$
- c. **Maximum and Minimum Values**
  - i. Take the partial derivatives to find critical points that could be local maximum/minimum
    - **1. Remember to check that the points you get are actually IN the domain and not on the edges or outside of the domain altogether**
  - ii. Take the second derivatives and use the second derivative test
    - 1. $D = f_{xx}(a, b) * f_{yy}(a, b) - [f_{xy}(a, b)]^2$
    - 2. If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum
    - 3. if $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum
    - 4. if $D < 0$, then $f(a, b)$ is a saddle point (neither a max nor min)
  - iii. Find the extreme values of f on the boundary of D, plugging them back into the equation to get the largest and smallest values
  - iv. The largest and smallest of all values of f(a,b) from both critical points inside the domain and extreme values on the edges of the domain are the absolute maximum and minimum
  - v. When trying to max an application problem, solve a function for z and substitute into the other so that you end up w/ one function f(x, y). Then take the partial derivatives, set them equal to 0, and find max/min.
- d. **LaGrange Multipliers**
  - i. LaGrange Multipliers are used to find the maximum and minimum values of f(x, y, z) subject to the constraint g(x, y, z) = k
  - ii. **Equation for one constraint $g(x_0, y_0, z_0)$**
    - 1. $\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$
  - iii. **Equation for two constraints, $g(x_0, y_0, z_0)$ and $h(x_0, y_0, z_0)$**
    - 1. $\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla \eta(x_0, y_0, z_0)$
  - iv. **Method of LaGrange Multipliers**
    - 1. Find all values of x, y, z, and λ such that $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = k$
    - 2. This involves solving a system of equations, and they're not always linear.

3. Evaluate f at all the points 9, y, z) that result from step 1. the largest of these values is the max value of f, and the smallest is the min value.
4. When there's two constraints, include the 2nd constraint, which will introduce one more variable, but also one more equation, so it's still a solvable system of equations

XI. Double Integrals

- a. Definition
  - i. $\iint f(x, y)dA = \lim (m, n \to \infty)\ \Sigma(i = 1, m)\ \Sigma(j = 1, n)\ f(x_{ij}^{*}, y_{ij}^{*})\ \Delta A$
  - ii. It's a double Riemann sum and is used as an approximation to the value of the double integral
  - iii. Split the area to be integrated into rectangles, add them all up separately, and multiply by the area of each rectangle. You should end up w/ m*n terms. ΔA is the area of each rectangle
- b. Integrate 1 over a certain domain and you'll get the area of the domain
- **c. Average Value**
  - i. $1/A(R) \iint f(x, y)dA$
  - ii. Integrate the x values on one integral, the y values on another
- d. All properties of single integrals generally also apply to double integrals
- e. **The Real Way** – Fubini's Theorem for Rectangles
  - i. $\int^{a}_{b}\int^{c}_{d} f(x, y)\, dy\, dx = \int^{c}_{d}\int^{a}_{b} f(x,y)\, dx\, dy$
- **f. Double Integrals over General Regions**
  - i. $\int^{a}_{b}\int^{g2(x)}_{g1(x)} f(x, y)\, dy\, dx$ – for curves bounded between two curves y = f(x)
  - ii. $\int^{c}_{d}\int^{h2(y)}_{h1(y)} f(x, y)\, dx\, dy$ – for curves bounded between two curves x = f(y)
- **g. Double Integrals in Polar Coordinates**
  - i. If f is continuous on a polar rectangle R given by $0 < a < r < b$, $\alpha < \theta < \beta$, $0 < \beta - \alpha < 2\pi$
  - ii. f(x, y) dA = $\int^{\alpha}_{\beta}\int^{a}_{b} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$
  - **iii. Don't forget the last r and dr**
- **h. Surface Area in 3D Space Using Double Integrals**
  - i. $A(S) = \iint(d)\ \text{Rad}\ (1 + (dz/dx)^2 + (dz/dy)^2)$
  - ii. To find the limits, reduce it to a 2d figure, set either x or y to a finite range from a to b, and integrate accordingly. For example, if x is set from 0 – 1, and y from 0 - x, then the integral would be $\int_0^1\int_0^x \text{Rad}\ (1 + (dz/dx)^2 + (dz/dy)^2)$
- i. When integrating over a rectangle, you can switch your limits of integration and integrate in any order you want. However, when you're not integrating over a rectangle, when you change the order of integration, be sure to adjust your limits accordingly, as they will be significantly different.
- **j. Draw a picture for every problem. It helps!**

XII. Triple Integrals

- a. Definition
  - i. $\iiint f(x, y, z)\, dV = \lim (l, m, n \to \infty)\ \Sigma(i = 1, l)\ \Sigma(j = 1, m)\ \Sigma(k = 1, n)\ f(x_{ijk}^{*}, y_{ijk}^{*}, z_{ijk}^{*})\ \Delta V$
- b. Integrate 1 over a certain domain and you'll get the volume of the domain
- **c. Fubini''s Theorem Over a Rectangular Box**
  - i. $\int^{s}_{r}\int^{d}_{c}\int^{b}_{a} f(x, y)\, dx\, dy\, dz$
  - ii. Feel free to change the order of integration any way you like
- **d. General Equation for Triple Integrals**
  - i. $\int_{a}^{b}\int_{g1(x)}^{g2(x)}\int_{u1(x,y)}^{u2(x,y)} f(x, y, z)\, dz\, dy\, dx$
  - ii. The limits of integration will have to be found by observation. Project the figure onto the xy plane. Dix x, get y, then get z. Do it in any order, but know to adjust limits accordingly

XIII. Memorize

- a. Identities
- b. Angle values
- c. Integration

Stop taking shortcuts, multiply everything out.